

On the S-Matrix of the Faddeev-Reshetikhin Model

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Abstract. The Faddeev-Reshetikhin model arises as a truncation of strings in $AdS_5 \times S^5$. Its two particle S-matrix should be obtained by diagonalizing its Hamiltonian. However this does not happen in a straightforward way. There is a Lorentz violating term in the Hamiltonian which prevents its plain diagonalization. We then find out the most general quartic Hamiltonian that can be diagonalized. It includes the bosonic Thirring model as well as another model which shares the same two particle S-matrix as the Faddeev-Reshetikhin model. We also find an one parameter family of interactions which lead to non-Hermitian Hamiltonians which have unitary S-matrices and are PT symmetric.

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INTRODUCTION

Ten years ago a daring conjecture [1, 2, 3] was made between two apparently disconnect theories. It was proposed that string theory in $AdS_5 \times S^5$ is dual to the $\mathcal{N} = 4$ supersymmetric gauge theory in four dimensions. Ten years later [4, 5], a huge amount of evidence now supports that claim and it seems we are heading to a proof of the conjecture. To show that the correspondence is true it was initially thought that we should construct all the correlation functions of the gauge theory and all the states of the string theory. This is a huge task but these were the first steps taken to show the conjecture. Even though the relation between the two theories is a duality between the weak and strong coupling regimes many successful results were obtained [4]. Reliable computations can be done in the weak coupling regime of both theories but the interpolating region was always difficult to be accessed. In the last five years, however, it was found that both theories have integrable sectors. The classical action of string theory in $AdS_5 \times S^5$ is integrable in the sense that an infinite set of non-local currents can be found [6]. On the other side, the problem of finding the anomalous dimensions of gauge invariant operators in the planar limit of the gauge theory can be mapped to the problem of diagonalizing a spin chain [7]. As a result, the energy spectrum of the string is conjectured to coincide with the spectrum of anomalous dimensions. Thanks to the integrability properties of both sides this was verified in several limits where they can be compared with amazing success [5]. Today we are here to celebrate the successes reached in these ten years.

String theory is integrable at the classical level but quantum integrability is still an open issue. One approach to treat the quantum case is to consider truncated strings in $AdS_5 \times S^5$. Of course, the quantization of truncated strings can at most be regarded as a laboratory to test its integrability properties because the frozen degrees of freedom are

not being properly taken into account. Surprisingly many sectors still show integrability at the quantum level. For instance, if the string is truncated to the $R \times S^3$ sector of $AdS_5 \times S^5$, with R being the time in AdS_5 and S^3 a subspace of S^5 , then the resulting model is an $SU(2)$ principal chiral sigma model which was analysed in detail by Faddeev and Reshetikhin [8]. They proposed that the Hamiltonian for the original sigma model as well as the nonultralocal Poisson brackets should be replaced by another Hamiltonian and a new Poisson bracket structure which lead to the same dynamical equations. After the quantization of this new system the original sigma model can be recovered by taking a certain limit. More recently, Klose and Zarembo [9] used standard field theoretical techniques to obtain its two particle S-matrix. If the false vacuum is chosen the calculations are strongly simplified since only bubble diagrams contribute to the S-matrix.

An alternative way to obtain the S-matrix for the Faddeev-Reshetikhin model is by a straight diagonalization of its Hamiltonian [10]. Amazingly, the quartic Hamiltonian for the Faddeev-Reshetikhin model cannot be diagonalized. The trouble is in a Lorentz violating term in the interaction Hamiltonian. This is very surprising since the calculation of the S-matrix by Klose and Zarembo [9] is straightforward. Even so, our results are consistent with the field theory results if we consider the Hamiltonian matrix elements of positive energy states only. This is completely consistent with the field theoretic calculation since it involves only the calculation of matrix elements. This leads to the conclusion, which had not been observed earlier in other integrable models, that while the diagonalization of a system leads to its S-matrix, having the S-matrix does not automatically imply diagonalizability of the system.

The lack of diagonalizability of the quartic Hamiltonian then lead us to determine the most general quartic Hamiltonian which can be diagonalized assuming integrability [10]. It includes the bosonic Thirring model, which is known to be integrable, as well as the bosonic chiral Gross-Neveu model, which to the best of our knowledge had not been studied earlier. Both models respect Lorentz invariance and, in fact, we find that the two systems share the same S-matrix. We show that this puzzling coincidence is due to a Fierz transformation so that both models are in fact equivalent. We also determined the most general quartic Hamiltonian, violating Lorentz invariance, which can be diagonalized and leads to the same S-matrix as that calculated by Klose and Zarembo for the Faddeev-Reshetikhin model. Furthermore, we found that in spite of the fact that the spectrum of this generalized family of Hamiltonians is real and the S-matrix is unitary, the Hamiltonian is not Hermitian but is PT symmetric. It falls in a class of quantum mechanical systems with a non Hermitian Hamiltonian but unitary S-matrix [11].

STRINGS IN $R \times S^3$

The action for a string in $AdS_5 \times S^5$ is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma (G_{\mu\nu}^{(AdS_5)} \partial^\alpha X^\mu \partial_\alpha X^\nu + G_{ab}^{(S^5)} \partial^\alpha Y^a \partial_\alpha Y^b). \quad (1)$$

The embedding coordinates in AdS are X^0, \dots, X^5 and those for S^5 are Y^1, \dots, Y^6 and they satisfy

$$(X^0)^2 + (X^1)^2 + \dots + (X^4)^2 - (X^5)^2 = R^2, \quad (2)$$

$$(Y^1)^2 + \dots + (Y^6)^2 = R^2, \quad (3)$$

The truncation to $R \times S^3$ is just given by $X^0 = t, Y^5 = Y^6 = 0$ and vanishing the remaining variables in AdS . The Virasoro constraints then reduce to

$$\Lambda_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}\Lambda_\gamma^\gamma = 0, \quad (4)$$

where

$$\Lambda_{\alpha\beta} = \delta_{ab}\partial_\alpha Y^a \partial_\beta Y^b - \partial_\alpha t \partial_\beta t, \quad a, b = 1 \dots 4. \quad (5)$$

To make connection with the $SU(2)$ principal chiral sigma model we now use the standard map of S^3 to $SU(2)$

$$g = \begin{pmatrix} Y^1 + iY^2 & Y^3 + iY^4 \\ -Y^3 + iY^4 & Y^1 - iY^2 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}, \quad (6)$$

where $\det g = 1$ thanks to the constraint on the embedding coordinates of S^5 . We then define the $SU(2)$ current in the usual way $J = g^{-1}dg$ so that the string equations of motion are

$$\partial_\alpha J^\alpha = 0, \quad \partial^2 t = 0, \quad (7)$$

while the Virasoro constraints reduce to

$$Tr(J_\pm^2) = -2(\partial_\pm t)^2. \quad (8)$$

We can also choose the gauge $t = \kappa\tau$ so that the Virasoro constraints simplify to $Tr(J_\pm^2) = -2\kappa^2$.

To make contact with the usual form of the Faddeev-Reshetikhin model we recall that J_\pm is a two by two matrix and can be expanded in terms of the Pauli matrices. We do that by introducing the variables \vec{S}_\pm as

$$J_\pm = i\kappa \vec{S}_\pm \cdot \vec{\sigma}. \quad (9)$$

Then the Virasoro constraints become $\vec{S}_\pm^2 = 1$ and the equations of motion reduce to

$$\partial_\mp \vec{S}_\pm \mp \kappa \vec{S}_+ \times \vec{S}_- = 0, \quad (10)$$

which is the familiar form of the Faddeev-Reshetikhin equations. As remarked in the introduction, this model has been solved using the inverse scattering method [8] and field theoretic techniques [9]. Our approach [10] is the diagonalization of the system.

The Faddeev-Reshetikhin equations can be obtained from the following action

$$S = \int d^2x [C_+(\vec{S}_-) + C_-(\vec{S}_+) + \frac{\kappa}{2} \vec{S}_+ \cdot \vec{S}_-], \quad (11)$$

where $C_{\pm}(\vec{S}_{\mp})$ are the usual Wess-Zumino terms

$$C_{\pm}(\vec{S}_{\mp}) = -\frac{1}{2} \int_0^1 d\xi \epsilon^{abc} S_{\mp}^a \partial_{\xi} S_{\mp}^b S_{\mp}^c. \quad (12)$$

To get a canonical kinetic term and unconstrained variables we change variables again to

$$\phi_{\pm} = \frac{S_{\pm}^1 + iS_{\pm}^2}{\sqrt{2}\sqrt{1+S_{\pm}^3}}, \quad S_{\pm}^3 = 1 - 2|\phi_{\pm}|^2, \quad (13)$$

so that the action takes the form

$$\begin{aligned} S = & \int d^2x \left[\frac{i}{2}(\phi_-^* \partial_+ \phi_- - \phi_- \partial_+ \phi_-^*) + \frac{i}{2}(\phi_+^* \partial_- \phi_+ - \phi_+ \partial_- \phi_+^*) + \kappa(|\phi_+|^2 + |\phi_-|^2) \right. \\ & \left. - \kappa \sqrt{(1 - |\phi_+|^2)(1 - |\phi_-|^2)} (\phi_+^* \phi_- + \phi_-^* \phi_+) - 2\kappa|\phi_+|^2|\phi_-|^2 \right]. \end{aligned} \quad (14)$$

It can be rewritten in a more compact Dirac like form as

$$S = \int d^2x [\bar{\phi} i\gamma^{\mu} D_{\mu} \phi - m\bar{\phi} \phi - g(\bar{\phi} \gamma^{\mu} \phi)^2 + \mathcal{O}(\phi^6)], \quad (15)$$

where

$$\phi = \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix}, \quad (16)$$

and $D_0 = \partial_0 - im - ig\bar{\phi}\phi$, $D_1 = \partial_1$; $m = \kappa$ and $g = \kappa/2$. Even though the action looks Lorentz invariant it is not so because of the derivatives. It is just convenient to write the action in this way. Notice also that the square root term was expanded up to fourth order in the fields.

DIAGONALIZATION OF THE HAMILTONIAN

To obtain the two particle S-matrix we need the Hamiltonian up to fourth order in the fields. From the action in the previous section we easily compute the Hamiltonian

$$H = \int dx : [-\bar{\phi} i\gamma^1 \partial_x \phi + m\bar{\phi} \phi + g(\bar{\phi} \gamma^0 \phi \bar{\phi} \phi) - (\bar{\phi} \gamma^{\mu} \phi)^2] :. \quad (17)$$

We assume the standard commutation relations $[\phi_{\alpha}(x), \phi_{\beta}^{\dagger}(y)] = \delta_{\alpha\beta} \delta(x-y)$ and choose the false vacuum $\phi(x)|0\rangle = 0$. We can go the true vacuum later on by filling all the negative energy states.

The two particle states can be written as

$$|k_1, k_2\rangle = \int dx_1 dx_2 \chi_{\alpha\beta}(x_1, x_2, k_1, k_2) \phi_{\alpha}^{\dagger}(x_1) \phi_{\beta}^{\dagger}(x_2) |0\rangle, \quad (18)$$

where $\chi_{\alpha\beta}(x_1, x_2) = \chi_{\beta\alpha}(x_2, x_1)$ is the wave function. For the free theory we can make use of the Dirac technology: define the two component solutions of the momentum space Dirac equation $u(k)$ and $v(k)$ as

$$(\not{k} - m)u(k) = 0, \quad (\not{k} + m)v(k) = 0, \quad (19)$$

with

$$u(k) = \sqrt{m} \begin{pmatrix} e^{\beta/2} \\ e^{-\beta/2} \end{pmatrix}, \quad v(k) = \sqrt{m} \begin{pmatrix} -e^{\beta/2} \\ e^{-\beta/2} \end{pmatrix}, \quad (20)$$

where β is the rapidity defined as usual by $k^0 = m \cosh \beta$ and $k^1 = m \sinh \beta$. We now look for eigenfunctions of the momentum with eigenvalue $k_1 + k_2$ and eigenfunctions of the Hamiltonian with eigenvalue $E = \sqrt{m^2 + k_1^2} + \sqrt{m^2 + k_2^2}$. They can be easily found to be

$$\chi_{\alpha\beta}^{(\pm)}(x_1, x_2, k_1, k_2) = e^{i(k_1 x_1 + k_2 x_2)} u_\alpha(k_1) u_\beta(k_2) \pm e^{i(k_1 x_2 + k_2 x_1)} u_\alpha(k_2) u_\beta(k_1). \quad (21)$$

We now turn to the interacting case. The ansatz is a superposition of the free solutions with the same eigenvalues for the momentum and energy as in the free case as usual in integrable models. We denote it by

$$\chi_{\alpha\beta}(x_1, x_2) = \chi_{\alpha\beta}^{(+)}(x_1, x_2) + \lambda(k_1, k_2) \varepsilon(x_1 - x_2) \chi_{\alpha\beta}^{(-)}(x_1, x_2), \quad (22)$$

with $\lambda(k_1, k_2)$ to be determined. Written in this form the two particle S-matrix takes the form $S = (1 - \lambda)/(1 + \lambda)$. Remarkably, there is no solution for the eigenvalue problem! The Hamiltonian is not diagonalizable.

Let us take a closer look at the interaction Hamiltonian which can be written as

$$H_I |k_1, k_2\rangle = \frac{g}{2} \int dx_1 dx_2 \delta(x_1 - x_2) V_{\alpha\beta\gamma\delta} \chi_{\gamma\delta}(x_1, x_2) \phi_\alpha^\dagger(x_1) \phi_\beta^\dagger(x_2) |0\rangle, \quad (23)$$

showing that the interaction is produced by a single Dirac delta interaction. Instead of solving the eigenvalue problem we could try to match the discontinuity produced by the delta function. Again, no solution was found. If the Lorentz violating term involving the γ^0 contribution in (17) is eliminated then a solution can be found. Removing the γ^0 term in the Hamiltonian reduces it to the known bosonic massive Thirring model and the solution found corresponds indeed to the two particle S-matrix of this relativistic model

$$S = \frac{1 - ig \coth(\beta_1 - \beta_2)/2}{1 + ig \coth(\beta_1 - \beta_2)/2}. \quad (24)$$

To get the S-matrix for the Faddeev-Reshetikhin model, which is not relativistic, we need only matrix elements between say positive-positive energy states. We do not need the knowledge of the intermediate states. So consider again the eigenvalue equation and take the inner product with a positive energy state and solve for $\lambda(k_1, k_2)$. Now a solution exists and we obtain

$$\lambda = ig \frac{\cosh \frac{\beta_1 + \beta_2}{2} - \cosh \frac{\beta_1 - \beta_2}{2}}{\sinh \frac{\beta_1 - \beta_2}{2}}, \quad (25)$$

which is indeed the correct form for λ to produce the two particle S-matrix for the Faddeev-Reshetikhin model. So, a diagonalizable Hamiltonian leads to a S-matrix but having a S-matrix does not automatically imply that the Hamiltonian is diagonalizable!

GENERAL QUARTIC HAMILTONIAN

At this point is natural to look for the most general quartic Hamiltonian which can be diagonalized, under the assumption of integrability, and its corresponding two particle S-matrix. We found a one parameter set of models

$$H_I = \int dx [\alpha(\bar{\phi}\gamma^\mu\phi)^2 + \beta\bar{\phi}\gamma^\mu\gamma^0\phi\bar{\phi}\gamma_\mu\phi], \quad (26)$$

with

$$\lambda = -i \left[\alpha \coth \frac{\beta_1 - \beta_2}{2} + \beta \frac{\cosh \frac{\beta_1 + \beta_2}{2}}{\sinh \frac{\beta_1 - \beta_2}{2}} \right], \quad S = \frac{1 - \lambda}{1 + \lambda}. \quad (27)$$

For $\alpha = g$ and $\beta = 0$ we regain the bosonic massive Thirring model discussed in the previous section. For $\alpha = -\beta = g$ we get a one parameter family of models with the same two particle S-matrix as the Faddeev-Reshetikhin model. Due to the Fierz identity

$$(\bar{\phi}\phi)^2 - (\bar{\phi}\gamma_5\phi)^2 = (\bar{\phi}\gamma^\mu\phi)^2, \quad (28)$$

the Hamiltonian (26) with $\beta = 0$ reduces to the bosonic chiral Gross-Neveu model and it shares the same S-matrix as the bosonic massive Thirring model. As far as we know, this has not been noticed before. This was verified explicitly

Notice that hermiticity is in trouble in the β term since $(\bar{\phi}\gamma^0\gamma^1\phi)^\dagger = -\bar{\phi}\gamma^0\gamma^1\phi$ is pure imaginary while $(\bar{\phi}\gamma^1\phi)^\dagger = \bar{\phi}\gamma^1\phi$ is real. Then the Hamiltonian is not hermitian for real β but its S-matrix is unitary! Quantum mechanical theories with non-Hermitian Hamiltonians which still have real energy are well known [11]. For these theories the requirement of hermiticity is replaced by the weaker requirement of being PT symmetric. In our case, the PT symmetry is given by

$$\begin{aligned} P : \quad & \phi(x, t) \rightarrow \eta_P \gamma^0 \phi(-x, t), \\ T : \quad & \phi(x, t) \rightarrow \eta_T C \gamma^0 \gamma^1 \phi(x, -t), \end{aligned} \quad (29)$$

and the Hamiltonian is PT symmetric if α and β are real. This is possibly the first example of an integrable field theory which has a non-Hermitian Hamiltonian, is PT symmetric, and has a unitary S-matrix.

CONCLUSION

We have studied in detail the Faddeev-Reshetikhin model which appears when string theory in $AdS_5 \times S^5$ is truncated to $R \times S^3$. Even though the S-matrix of the theory has been calculated using field theoretic methods, diagonalization of the Hamiltonian is essential to carry out the Bethe ansatz analysis. We found that the quartic Hamiltonian for

this model is not diagonalizable in the two particle sector. We found that the difficulty is in the term in the interaction Hamiltonian violating Lorentz invariance. If we take the inner product of the eigenvalue equation with positive energy states the problematic term disappears leading to the correct S-matrix element calculated earlier. Further investigation shows that the interaction generates intermediate states that are orthogonal to the positive energy out states and, therefore, cannot be observed in the S-matrix calculation but are relevant in the diagonalization of the system. To the best of our knowledge, this is a new feature that has not been observed earlier in the study of integrable systems. It follows, therefore, that while the diagonalization of a Hamiltonian leads to the S-matrix of the theory, the knowledge of the S-matrix element by other means does not automatically guarantee the diagonalizability of the Hamiltonian of the system. We have also determined the most general Hamiltonian with quartic interactions that can be diagonalized as well as its S-matrix. Among various special cases, it also includes a generalized Hamiltonian that can be diagonalized with the same S-matrix as that calculated by Klose and Zarembo. We showed that although this general Hamiltonian leads to a real spectrum and a unitary S-matrix, it is not Hermitian. However, it is PT symmetric and the wave functions are also invariant under PT transformations. As a result, the theory is in an unbroken phase of PT symmetry which is the reason for the reality of the spectrum as well as the unitarity of the S-matrix.

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